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# STABILITY AND REGULARIZATION OF SOLUTIONS OF NONLOCAL INITIAL-BOUNDARY VALUE PROBLEMS

Sapayev Oybek Azadboy ugli

Master's student in mathematics at the International University of Asia

**Abstract.** We study an evolution problem in which the classical initial condition is replaced by the nonlocal condition $\alpha u(0) + u(T) = g$. We obtain a uniqueness criterion and sharp stability estimates in terms of the parameter $\alpha$. We then use the nonlocal problem as a regularization method for the ill-posed backward problem and prove a Hölder-type convergence rate for $t > 0$ and a logarithmic rate at $t = 0$. All results are proved analytically.

**Keywords:** nonlocal condition; stability; ill-posed problem; quasi-boundary value method; regularization.

## 1. Introduction

Nonlocal conditions, which link the values of a solution at different points or moments, arise in heat conduction, diffusion and vibration problems where the initial state cannot be measured directly. Problems of this type go back to Ionkin [1] and were studied for hyperbolic equations by Kozhanov and Pulkina [2]. Conditions that are nonlocal in time are also closely related to regularization theory. In the quasi-boundary value method [3, 4], the ill-posed backward problem $u(T) = g$ is replaced by a well-posed nonlocal problem $\alpha u(0) + u(T) = g$ with a small parameter $\alpha > 0$. This method complements the classical regularization methods of Tikhonov [5] and of quasi-reversibility [6].

The aim of this work is to describe exactly for which $\alpha$ the nonlocal problem is stable, to obtain explicit stability constants, and to derive convergence rates of the corresponding regularization method.

2. Problem statement and method

Let H be a Hilbert space and let A be a positive self-adjoint operator in H with compact inverse. Denote its eigenvalues by $0 < \lambda_1 \le \lambda_2 \le \ldots \to \infty$ and its orthonormal eigenvectors by $X_n$. A typical example is $A = -d^2/dx^2$ on $(0, \pi)$ with Dirichlet conditions, where $\lambda_n = n^2$ and $X_n = (2/\pi)^{1/2} \sin nx$. For $T > 0$, $\alpha \in \mathbb{R}$ and $g \in H$, consider the problem

The method of investigation is the spectral (Fourier) method. Write $z_n = e^{-\lambda_n T} \in (0, 1)$ and $g_n = (g, X_n)$. Expanding u(t) in the eigenvectors, we obtain $u_n(t) = u_n(0)e^{-\lambda_n t}$ and $(\alpha + z_n)u_n(0) = g_n$. Hence, if $\alpha + z_n \neq 0$ for all n,

Stability is thus governed by the multipliers in (2). They are estimated with the weighted arithmetic–geometric mean inequality: for $\theta \in [0, 1]$, $\alpha > 0$ and $z > 0$,

3. Main results

**Theorem 1 (uniqueness and stability).** Let $E = \{-e^{-\lambda_n T} : n \in \mathbb{N}\}$.

Problem (1) has at most one solution if and only if $\alpha \notin E$.

If $\alpha > 0$, then for every $g \in H$ and every $t \in [0, T]$

If $\alpha < 0$ and $\alpha \notin E$, then $\|u(t)\| \le d^{-1}\|g\|$, where $d = \inf_n |\alpha + z_n| > 0$.

If $\alpha = 0$ (the backward problem), no estimate $\|u(0)\| \le C\|g\|$ holds.

*Proof.* (a) If $\alpha = -z_m$, then $u(t) = e^{-\lambda m t}X_m$ solves (1) with $g = 0$. Otherwise (2) shows that zero data give $u \equiv 0$. (b) Since $e^{-\lambda_n t} = z_n^{\theta}$, inequality (3) gives $z_n^{\theta}/(\alpha + z_n) \le c(\theta)\alpha^{\theta-1}$, and (4) follows from Parseval's identity. (c) Since $z_n \to 0$, we have $|\alpha + z_n| \to |\alpha| > 0$, so $d > 0$; moreover $e^{-\lambda_n t} \le 1$. (d) For $g = X_n$ we get $\|u(0)\| = e^{\lambda_n T} \to \infty$. ■

Estimate (4) shows that the problem is well posed for every $\alpha > 0$, but the constant grows like $\alpha^{-1}$ at $t = 0$ as $\alpha \to 0$. This is precisely the mechanism that makes the nonlocal problem a regularization of the backward problem.

**Theorem 2 (Hölder rate for t > 0).** Let u be the solution of the backward problem $u' + Au = 0$, $u(T) = g$, with $\|u(0)\| \le E$. Let $\|g^{\delta} - g\| \le \delta$, and let $u_{\alpha}^{\delta}$ solve (1) with data $g^{\delta}$. Then for $0 < t \le T$

With $\alpha = \delta/E$ this gives

*Proof.* The noise term is bounded by (4). For exact data, $g_n = z_n u_n(0)$, so the n-th coefficient of the error equals $-\alpha e^{-\lambda_n t} u_n(0)/(\alpha + z_n)$. By (3), its multiplier satisfies $\alpha z_n^{\theta}/(\alpha + z_n) \le \alpha^{\theta}$. ■

Rate (6) coincides with the classical conditional stability estimate $\|u(t)\| \le \|u(0)\|^{1-\theta}\|u(T)\|^{\theta}$ given by logarithmic convexity [7]. Therefore, the regularization is order-optimal for $t > 0$.

**Theorem 3 (logarithmic rate at t = 0).** Under the conditions of Theorem 2, assume in addition that $\|Au(0)\| \le E_1$, and let $0 < \delta < 1$. Then for $\alpha = \delta^{1/2}$

*Proof.* The noise term is at most $\delta/\alpha$. The bias coefficients are $\alpha|u_n(0)|/(\alpha + z_n)$. For $\lambda_n \le \Lambda$ they are bounded by $\alpha e^{\Lambda T}|u_n(0)|$, and for $\lambda_n > \Lambda$ by $\lambda_n|u_n(0)|/\Lambda$. Hence the bias is at most $\alpha e^{\Lambda T}E + E_1/\Lambda$. The choice $\Lambda = \ln(1/\alpha)/(2T)$ gives

and $\alpha = \delta^{1/2}$ yields (7). ■

**Example.** Let $A = -d^2/dx^2$ on $(0, \pi)$ and $T = 1$. For $\alpha = -e^{-1} \in E$, the function $u = e^{-t}\sin x$ satisfies $\alpha u(0) + u(1) = 0$, so uniqueness fails, in agreement with Theorem 1(a). For $\alpha > 0$, estimate (4) at $t = 0$ gives $\|u(0)\| \le \alpha^{-1}\|g\|$.

**Remark (hyperbolic case).** For the wave equation $u'' + Au = 0$ with conditions that are nonlocal in time, the multipliers contain denominators such as $1 - \cos(\lambda_n^{1/2}T)$. These do not tend to a nonzero limit, so stability depends on the Diophantine properties of T rather than on the sign of $\alpha$. Such small-denominator problems are studied in [8]; for them, spectral cut-off is the natural regularization [9].

4. Conclusion

For the parabolic nonlocal problem $\alpha u(0) + u(T) = g$, uniqueness holds exactly when $\alpha \notin \{-e^{-\lambda_n T}\}$. Stability holds for every such $\alpha \neq 0$, with the explicit constant $\alpha^{t/T-1}$ for $\alpha > 0$. Used as a quasi-boundary regularization of the backward problem, the nonlocal problem gives the order-optimal Hölder rate $E^{1-t/T}\delta^{t/T}$ for $t > 0$ and a

logarithmic rate at t = 0. These results provide explicit rules for choosing the regularization parameter α from the noise level δ.

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